TURBULENT WAKE IN A STRATIFIED MEDIUM

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The problem of determining the form of the turbulent wake being formed behind a self-propelled body in a medium with density varying in the direction of the effect of gravity is considered. The schematic picture of wake development behind a moving object is the following: Initially, diffusion is identical in all directions, and the wake broadens symmetrically; diffusion becomes strongly anisotropic with recession from the object, it diminishes in the vertical direction under the effect of gravity, and the wake becomes flattened; turbulent mixing within the wake results in the production of a more homogeneous density distribution within the volume occupied by the wake than in the surrounding medium; such a fluid volume turns out to be removed from the equilibrium state and tends to return to the equilibrium state under the effect of gravity; collapse of the wake occurs accompanied by its further expansion in the horizontal direction and the excitation of internal waves.

The problem of the first stage of wake developemnt (prior to collapse), i.e., the problem of turbulent diffusion in a stratified medium, is considered herein. The medium itself is at rest. Equations obtained in [1], which permit the description of anisotropic diffusion, are the basis for the description of the diffusion process. The standard simplifications used in problems involving the propagation of a turbulent wake (reflecting the experimental observation that the free turbulent zones are relatively narrow) are used. Molecular diffusion is not taken into account. It is considered that density pulsations are small and, hence, should be taken into account only in members containing the acceleration of gravity [2].

The z axis is directed upward along the direction of the effect of gravity, and the x axis in the motion direction. The picture of the flow in the coordinate system connected with the body is stationary. The free stream velocity U_0 is assumed to be considerably greater than the velocity component in the wake. The scales along the axes l_x, l_y, l_z and the characteristic values of the pulsating velocities are distinct. The above-mentioned circumstances permit introduction of simplifications which are utilized in deriving the boundary layer equations. Moreover, the following simplifications are introduced: a) third-order moments corresponding to diffusion processes are neglected in all the equations; this is the customary assumption in the wake problem, otherwise it is necessary to make an assumption on the relation between the third-order moments and the average stream characteristics; b) constancy of the scale of turbulence L and the pulsating motion energy E in the wake cross section is assumed so that they vary only as a functions of the longitudinal coordinate.

The change in E across the wake can be considered additionally.

The system of equations is

$$U_0 \frac{\partial U}{\partial x} = -\frac{\partial}{\partial y}\langle u_x' u_y'\rangle - \frac{\partial}{\partial z}\langle u_x' u_z'\rangle$$

$$\langle u_x' u_y'\rangle = -\frac{\tau}{2}\left[\langle u_y'^2\rangle \frac{\partial U}{\partial y} + \langle u_y' u_z'\rangle \frac{\partial U}{\partial z} + U_0 \frac{\partial}{\partial x}\langle u_x' u_y'\rangle\right]$$

$$\langle u_x' u_z'\rangle = -\frac{\tau}{2}\left[g \frac{\langle \rho' U_x'\rangle}{\rho} + \langle u_z' u_y'\rangle \frac{\partial U}{\partial y} + \right.$$
$$\left. + \langle u_z'^2\rangle \frac{\partial U}{\partial z} + U_0 \frac{\partial}{\partial x}\langle u_x' u_z'\rangle\right]$$

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$$\langle u_y' u_z' \rangle = -\frac{\tau}{2}\left[g\frac{\langle \rho' u_y' \rangle}{\rho} + U_0 \frac{\partial}{\partial x}\langle u_y' y_z' \rangle\right]$$

$$\langle \rho' u_x' \rangle = -\frac{2}{3}\tau\left[\langle u_x' u_z' \rangle \frac{d\rho}{dz} + \langle \rho' u_z' \rangle \frac{\partial U}{\partial z} + U_0 \frac{\partial}{\partial x}\langle \rho' u_x' \rangle\right]$$

$$\langle \rho' u_y' \rangle = -\frac{2}{3}\tau\left[\langle u_y' u_z' \rangle \frac{d\rho}{dz} + U_0 \frac{\partial}{\partial x}\langle \rho' u_y' \rangle\right]$$

$$\langle \rho' u_z' \rangle = -\frac{2}{3}\tau\left[\langle u_z'^2 \rangle \frac{d\rho}{dz} + U_0 \frac{\partial}{\partial x}\langle \rho' u_z' \rangle\right]$$

$$\langle u_y'^2 \rangle = \frac{E}{3} - \frac{\tau}{2} U_0 \frac{\partial}{\partial x}\langle u_y'^2 \rangle$$

$$\langle u_z'^2 \rangle = \frac{E}{3} - \tau g \frac{\langle \rho' u_z' \rangle}{\rho} - \frac{\tau}{2} U_0 \frac{\partial}{\partial x}\langle u_z'^2 \rangle$$

$$U_0 \frac{\partial E}{\partial x} + \left\{\langle u_x' u_y' \rangle \frac{\partial U}{\partial y} + \langle u_y' u_z' \rangle \frac{\partial U}{\partial z}\right\} = -\frac{E}{\tau} - g\frac{\langle \rho' u_z' \rangle}{\rho}$$

$$\xi_0 L\left\{\langle u_x' u_y' \rangle \frac{\partial U}{\partial y} + \langle u_x' u_z' \rangle \frac{\partial U}{\partial z}\right\} + U_0 \frac{\partial}{\partial x} LE = -\alpha_s \frac{EL}{\tau} - 2\alpha g \frac{\langle \rho' u_z' \rangle}{\rho} L$$

$$\tau = ALE^{-1/2}$$

The quantities { } are averaged over the wake cross section; $A = 3.86$, $\alpha_s = 0.8$ [1], α are empirical constants.

As is done in the case of wake diffusion in a homogeneous medium [3, 4], the solution is sought in the form of power-series dependences on the longitudinal coordinate.

Estimates (members containing the derivatives with respect to x with the factor U_0 are not small quantities and, since $\tau \sim x$, taking them into account results in small corrections) can be obtained on the basis of the system of equations

$$\langle \rho' u_y' \rangle \cong -{}^2/_3\,\tau \langle u_y' u_z' \rangle \frac{d\rho}{dz}, \quad \langle \rho' u_z' \rangle \cong -{}^2/_3\,\tau \langle u_z'^2 \rangle \frac{d\rho}{dz}$$

$$\langle \rho' u_x' \rangle \cong -{}^2/_3\,\tau \langle u_x' u_z' \rangle \frac{d\rho}{dz} + {}^4/_9\,\tau^2 \langle u_z'^2 \rangle \frac{d\rho}{dz}\frac{\partial U}{\partial z}$$

$$\langle u_z'^2 \rangle \cong \frac{E}{3[1 + {}^2/_3\tau^2 N^2]}, \quad \langle u_y'^2 \rangle \cong \frac{E}{3}, \quad N^2 = g\left|\frac{1}{\rho}\frac{d\rho}{dz}\right|$$

The quantity $\langle u_y' u_z' \rangle \cong -{}^1/_2\,\tau^2 \langle u_y' u_z' \rangle N^2$, i.e., in the approximation under consideration, is

$$\langle u_y' u_z' \rangle = 0, \quad \langle u_x' u_y' \rangle = -\varepsilon \frac{\partial U}{\partial y}, \qquad \langle u_x' u_z' \rangle = -\varepsilon\varphi(\tau N)\frac{\partial U}{\partial z}$$

$$\varepsilon = \frac{\tau E}{6}, \quad \varphi(\tau N) = \frac{1 - {}^4/_9\tau^2 N^2}{(1 + {}^1/_3\tau^2 N^2)(1 + {}^2/_3\tau^2 N^2)}$$

The following equations are obtained for U, E, L

$$U_0 \frac{\partial U}{\partial x} = \frac{\partial}{\partial y}\left[\varepsilon\frac{\partial U}{\partial y}\right] + \frac{\partial}{\partial z}\left[\varepsilon\varphi\frac{\partial U}{\partial z}\right]$$

$$U_0 \frac{\partial E}{\partial x} - \varepsilon\left\{\left(\frac{\partial U}{\partial y}\right)^2 + \varphi\left(\frac{\partial U}{\partial z}\right)^2\right\} = -\frac{E}{\tau}\,\frac{1 + {}^8/_9\tau^2 N^2}{1 + {}^2/_3\tau^2 N^2}$$

$$U_0 \frac{\partial LE}{\partial x} - \xi_0 L\varepsilon\left\{\left(\frac{\partial U}{\partial y}\right)^2 + \varphi\left(\frac{\partial U}{\partial z}\right)^2\right\} = -\alpha_s \frac{EL}{\tau}\,\frac{1 + \tau^2 N^2({}^2/_3 + {}^4/_9\,\alpha/\alpha_s)}{1 + {}^2/_3\tau^2 N^2}$$

$$\tau = ALE^{-1/2}$$

The approximate condition [4]

$$\int U y^2\,dy\,dz = \text{const}, \qquad \int U z^2\,dy\,dz = \text{const}$$

must be added to these equations, which yields in the axisymmetric case

$$\int U r^3\,dr = \text{const}$$

for a wake with zero momentum, and

$$\int U\,dy\,dz = \text{const}$$

for a wake with constant momentum.

Let us introduce the following dependences:

$$U(0, 0, x)/U_0 \sim (x')^{-\alpha}, \quad E' = E/E_0 = (x')^{-\beta}, \quad L/L_0 = (x')^{\gamma}, \; l_y/l_y^{\circ} = (x')^{\delta}$$
$$l_z/l_z^{\circ} = (x')^{\varepsilon}, \quad x_k' = x_k/d, \; l_k' = l_k/d$$

where E_0, L_0, l_y°, etc. are the characteristic values of the quantities, and d is the body diameter. Then

$$\tau' = \tau/\tau_0 = (x')^{\gamma + 1/2\beta}, \quad \tau' E' = (x')^{\gamma - 1/2\beta}$$

From a comparison between the members in the equations, we have

$$\beta = 2 - 2\delta, \; \alpha = 1 - \delta, \; \gamma = \delta$$

At large distances the wake behaves as a plane, and this is because the scale l_z' tends to a constant value because of the rapid decrease in the value of $\langle u_z'^2 \rangle$ as x' grows. In order to see this, let us integrate the equation of motion

$$\frac{\partial U}{\partial x'} = \frac{\partial}{\partial y'}\left[\varepsilon' \frac{\partial U}{\partial y'}\right] + \frac{\partial}{\partial z'}\left[\varepsilon' \varphi \frac{\partial U}{\partial z'}\right], \; \varepsilon' = \varepsilon_0 \tau' E', \quad \varepsilon_0 = \frac{\tau_0 E_0}{6 U_0 d}$$

To find the solution of this equation for given initial values $U(0, y', z')$, we can utilize the Fourier transform method [5]

$$U^{\circ}(x', k_y, k_z) = \frac{1}{(2\pi)^2} \int U(x', y', z') \exp[-ik_y y' - ik_z z']\, dy'\, dz'$$

The equation becomes

$$\frac{dU^{\circ}}{dx'} = -U^{\circ} \varepsilon' (k_y^2 + k_z^2 \varphi)$$

and its solution will be

$$U^{\circ}(x', k_y, k_z) = U^{\circ}(0, k_y, k_z) \exp\left[-k_y^2 \int_0^{x'} \varepsilon'\, dx' - k_z^2 \int_0^{x'} \varepsilon' \varphi\, dx'\right]$$

Furthermore, applying the method elucidated in the paper cited, we obtain the solution as

$$U(x', y', z') = \frac{1}{\pi} \int U(0, y'', z'') \exp\left[-\left(\frac{y'' - y'}{l_y'}\right)^2 - \left(\frac{z'' - z'}{l_z'}\right)^2\right] \frac{dy''\, dz''}{l_y' l_z'}$$

where

$$l_y' = \left[4 \int_0^{x'} \varepsilon'\, dx'\right]^{1/2}, \quad l_z' = \left[4 \int_0^{x} \varepsilon' \varphi\, dx'\right]^{1/2}$$

The vertical scale l_z' tends to a constant value which passes through the maximum (because of the shape of the function $\varphi(x')$). The maximum corresponds to the value of the coordinate

$$m^* = \tau_0 N x^* / d$$

and equals

$$\max l_z' = \left[\frac{4}{\tau_0 N} \int_0^{m^*} \frac{\varepsilon' (1 - 4/9\,\zeta^2)\, d\zeta}{(1 + 2/3\,\zeta^2)(1 + 1/3\,\zeta^2)}\right]^{1/2}$$

For a wake with constant momentum the initial velocity distribution can be given in the form of a δ function

$$\int U(0, y'', z'')\, dy''\, dz'' = U_0 D$$

and the velocity distribution is

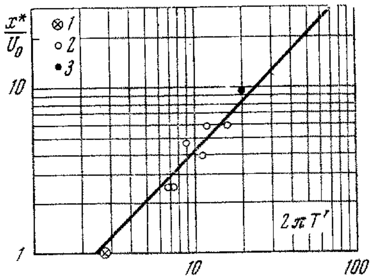

Fig. 1

$$U(x', y', z') = \frac{U_0 D}{\pi l_y' l_z'} \exp\left[-\left(\frac{y'}{l_y'}\right)^2 - \left(\frac{z'}{l_z'}\right)^2\right]$$

For large values of the coordinate x' we obtain

$$U(x, 0, 0) \sim x^{-1/2}, \quad E \sim x^{-1}, \quad L \sim l_y \sim x^{1/2}$$

In the case of a wake with zero momentum, an estimate can be obtained for the value of the velocity on the wake axis for large values of x'. Also assuming that $(l'_z)^2 \gg 1$

$$U(x', 0, 0) \cong \frac{-1}{\pi l_y' (l_z')^3} \int U(0, y'', z'')(z'')^2 dy'' dz''$$

we obtain

$$U(x, 0, 0) \sim x^{-1/2}, \quad E \sim x^{-1}, \quad L \sim l_y \sim x^{1/2}$$

In order to obtain a representation of the picture of wake development, let us utilize dependences obtained for large values of x in the integrands for l_y' and l_z'. Then

$$l_y' = \left[\frac{4\varepsilon_0}{2-\beta}\right]^{1/2} (x')^\delta = (4\varepsilon_0)^{1/2} (N\tau_0)^{-1/2} (N\tau_0 x')^{1/2}$$

$$l_z' = (4\varepsilon_0)^{1/2} (N\tau_0)^{-1/2} \left[\int_0^m \frac{(1 - {}^4/_9\, \zeta^2)\, d\zeta}{(1 + {}^2/_3\, \zeta^2)(1 + {}^1/_3\, \zeta^2)}\right]^{1/2}$$

The distance to m* includes the initial stage of wake development in which turbulent diffusion can be considered the predominant effect. The diminution of the vertical wake dimension obtained and the subsequent picture of its development require further analysis. The solution obtained permits establishment of a similarity criterion characterizing the length of the initial stage of wake development prior to collapse

$$N\tau_0 x^* / d = \text{const} \quad \text{or} \quad A\, \frac{L_0}{d} \frac{U_0}{E_0^{1/2}} \frac{x^* N}{U_0} = \text{const}$$

With respect to the factors L_0/d and $U_0/E_0^{1/2}$, it can be assumed that they depend slightly on the value of the Reynolds number and can be assumed constant with a good enough approximation. The constancy of the ratio L_0/d in jets and wakes is customarily assumed in analyzing these problems. Test data in [6] indicates the constancy of the quantity $U_0/E_0^{1/2}$ in some range of values of the Reynolds number. Therefore, the similarity criterion is the proportionality of the quantity x^*/U_0, i.e., the time after passage of the object prior to collapse of the wake, to the period of the natural oscillations of the medium

$$T' = N^{-1} = \left[g \left|\frac{1}{\rho} \frac{d\rho}{dz}\right|\right]^{-1/2}$$

This deduction corresponds to test data obtained on models (Fig. 1). The data are here marked as follows: 1) ⊗ from [7], 2) ○ from [9], 3) • from [8].

A comparison with experimental results yields the value

$$\frac{x^*}{U_0 T'} = 2.6$$

The relationship

$$\max l_z' \sim \left[\frac{T'}{\tau_0}\right]^{1/2} \left[\frac{\tau_0 E_0}{U_0 d}\right]^{1/2} \sim \left[\frac{T' U_0}{d}\right]^{1/2}$$

is obtained for the maximum value of the vertical scale of the wake.

The coefficient of wake flattening increases with distance and can reach a significant magnitude

$$n = \frac{l_y'}{l_z'} = m^{1/2} \left[\int_0^m \varphi(\zeta)\, d\zeta\right]^{-1/2} \sim (x')^{1/2} (N^2 \tau_0^2)^{1/4}$$

The motion of models in a stratified medium was studied in [7, 8]. The test conditions presented below were different, and the values of the ratio between the vertical dimension of the wake and the model diameter turned out to be almost identical in these tests. It is seen from the solution obtained that this agreement corresponds to the conditions existing.

According to [7] and [8] the test data are

$$U_0\,\text{cm/sec} = 45,\ 60;\ d\,\text{cm} = 2.2,\ 15;\ |d\rho_1/dz|\,\text{g}\cdot\text{cm}^{-4} = 0.0052,\ 0.0001;$$
$$T'\,\text{sec} = 0{,}438,\ 3.16;\ [T'U_0/d]^{1/2} = 3.0,\ 3.6$$

i.e., the values of l'_z should be sufficiently close together.

On the basis of a comparison with existing results, the deduction can be made that the equations utilized permit obtaining a solution yielding a satisfactory picture of wake development in the initial stage.

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